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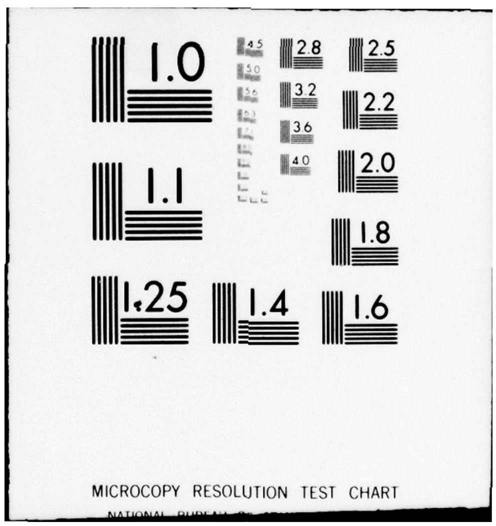
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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report discusses evaluation of expected steady state system availability when there is parallel redundancy and a pool of spares. The pool may support more than one system. Exact results are obtained for both warm standby and cold standby cases, assuming component installation time is 0. Approximations are suggested for treating non-zero installation time.

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PREFACE

As this paper was awaiting publication, the author realized that the steady state difference approach could also have been applied to the case of non-zero installation times to get exact answers. For some problems, but not all, the approach could serve as the basis of solution algorithms.

A future paper will pursue this line of approach, giving the details, and using the difference equations approach to evaluate the approximations provided in the paper.

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I. INTRODUCTION

1.1 Problem

This paper reports progress to date in evaluating expected steady state system availability in a single component (type) system with parallel redundancy, when there is a pool of spare components. There may be several systems supported by this same pool. The pool is managed by a supplier following an (S,S-1) inventory policy under continuous review.

The basic sequence of events is: component fails; spare component is obtained, immediately if one is available; component is installed. Because of redundancy, the system need not be down while this is occurring. If the pool of spares is not colocated with the systems, ship time is conceptually considered part of installation time. When a component fails, a good component is made available to the pool manager a resupply time later. This time represents time to repair the component, or order it if it is not repairable. For ease of exposition, we assume an unlimited number of repair facilities.

Two alternative basic premises are considered:

"Cold Standby" - Component only fails when it is being used, and only one of the parallel components is used at a time.

"Warm Standby" - All of the parallel components are in use and subject to failure at any given time, at the same rate.

1.2 Accomplishments Relative to Earlier Work

Surprisingly, considering the important practical applications, the problem addressed in this paper has received relatively little attention in the literature. Forry [3] reports on a model, ACCLOGTROM, with antecedents dating back to the 1960's, which not only evaluates system availability under more general system structures than are considered here, but optimizes inventory investment as well. However, this model rests on an assumption which is incorrect and dangerous, as discussed in Section 3.5. Similarly, Bein [2] considers evaluation of system availability in a system

with redundancy and supply, and allows for partial degradation, but his results do not rest on a rigorous foundation and could now be recast in more precise form.

Amongst more rigorous expositions, Gopalan's article [4] comes closest to the subject treated here. With proper interpretation of his work, a special case of our work corresponds to a special case of his; namely, the case of only one system supported by the supply pool and zero installation time. Gopalan assumes only one repair facility. Natarejan and Rao [3] allow multiple repair facilities, but in other ways their article is not as general as Gopalan's, and cannot be used to model warm standby with a supplier. Both the articles cited are representative of a number which have appeared, primarily in Operations Research, pertaining to availability of systems with redundancy and repair.

To summarize the contribution of this paper: for the first time exact results are developed, *for both cold* standby and warm standby, when installation time is zero, but there is more than one system supported by a pool manager. Approximations are developed for non-zero installation times. It is our expectation that the results will provide the basis for constituting a more accurate alternative to ACCLOGTROM, as they are readily integrated into a multi-component (type), multi-echelon context. It is our hope that the approximations presented will serve as something to "shoot at" in developing still more accurate approaches to treating problems when installation times are significant.

1.3 Notation

General

λ - failure rate for

a system	Cold Standby
a component	Warm Standby

μ - supplier's mean resupply time

n - number of systems supported by supplier

c - number of components per system

s - supplier's stockage parameter

B - number of backorders at random point in time

B_I - number of backorders attributable to a system I, chosen at random (B_I = B if n = 1)

U - probability a system is unavailable, i.e. down, in steady state

F(x;v) - cumulative Poisson with mean v.

Birth-Death Process

k - number of components in resupply, i.e. in repair or on order

λ_k - arrival rate when process is in state k

μ_k - departure rate when process is in state k

p_k - probability process is in state k at time t, calculated as limit as $t \to \infty$

Renewal Oriented

MTBF - mean time between failure

MLDT - mean logistics down time

MTI - mean time to install a component

II. "COLD STANDBY"

2.1 Exact Formulas When MTI = o

This problem may be modelled as a birth death process as has been shown in the literature. Component failures are births, and a death occurs after a service, i.e. a resupply, time. Sherbrooke [9] shows that a finite state birth-death process with state dependent arrival and service rates, exponential interarrival time, and general service distribution can be modelled exactly, and in fact state probabilities are identical to those for exponential service.

One System Supported. If there is only one system supported, we have a special case modelled as an M/M/c+s/c+s queue. In general queueing literature it is interpreted to mean that once c+s components are in resupply, any future demands are "lost," i.e., do not affect the state of the process. We may interpret it as meaning that if there are c+s units in resupply, the system is down and additional demands cannot occur. For this process, cf [5],

$$(2.1.1) \qquad p_k = \frac{(\lambda/u)^k/k!}{\sum_{i=o}^{c+s} (\lambda/u)^i/i!} \qquad k = o \text{ to } c+s$$

$$U = p_{c+s}$$

Multiple Systems Supported: Cannibalization. For n > 1, things get more complex unless we assume cannibalization, using the word in its most general sense to mean we manage our component backorders so as to have the fewest possible systems down. With cannibalization we retain the properties of a simple birth-death process; i.e., the state of the process is determined by the number in resupply.

In particular, if there are up to s+n(c-1) units in resupply, the number of backorders does not exceed (n)(c-1) and under cannibalization there are no systems down; each system will have (c-1) backorders. If there are s+(n)(c-1) + 2 units in resupply, two systems will be down, so unavailability given k = s + (n)(c-1) + 2 is 2/n. More generally,

$$U = \sum_{k=s+n(c-1)+1}^{s+nc} p_k \frac{[\text{Number of Systems Down in State } k]}{n}$$

$$(2.1.2) \quad = \sum_{k=s+n(c-1)+1}^{s+nc} \frac{(p_k)[k - s - n\,(c-1)]}{n}$$

where [cf Gross and Harris]

$$p_k = p_o \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}$$

$\lambda_k = n\lambda$ for $k = 0$ to $s+(n)(c-1)$

$\lambda_k = (n-m)\lambda$ for $k = s+(n)(c-1) + m \qquad m = 1$ to $n-1$

$\mu_k = k\mu$ for all k

p_o is determined so $\sum_{k=o}^{s+nc} p_k = 1$

<u>Multiple Systems Supported: No Cannibalization</u>. Let us clarify what is involved with a case where $n = 2$, $c = 2$. With cannibalization,

$\lambda_k = 2\lambda \quad ; \quad k = 0$ to $s+2$

$\lambda_k = \lambda \quad ; \quad k = s+3$

$\mu_k = k\mu \quad ; \quad$ all k

Without cannibalization, we must distinguish $p_{s+2,a}$ and $p_{s+2,b}$ where "a" denotes the event one system is down and "b" the event no systems are down. (For $k \neq s+2$ we have no indeterminancy). Results then depend on the pool manager's issue policy.

The policy of most interest is what we label the "smart" policy in which backorders are filled in the sequence designed to minimize number of systems down. A set of steady state difference equations can be defined in the usual form:

$$0 = [\text{Probability of Leaving State } k] + [\text{Probability of Entering State } k]$$

and the steady state values for all states obtained. In particular,

$$(2.1.3a) \qquad 0 = -[\lambda+(\mu)(s+2)]\ p_{s+2,a} + 2\lambda/2\ \ p_{s+1}$$

$$(2.1.3b) \qquad 0 = -[2\lambda + (\mu)(s+2)]\ p_{s+2,b} + 2\lambda/2\ \ p_{s+1} + (\mu)(s+3)\ p_{s+3}$$

Equations (2.1.3) reflect these observations: when one component is backordered the next demand is equally likely to come from either system. If 3 components are backordered, and one comes back from supply, it will be placed on the down system.

Appendix I gives the full set of equations and develops the solution. The analysis can in principle be extended to general n,c. For example, for c > 2, a smart policy would be to satisfy a backorder on the system with the maximum number of components backordered,requiring further elaboration on the state space. The analysis gets very tedious, but is sufficiently structured to offer expectation of feasible implementation by appropriate computer software.

Bounds on system availability are readily obtained. Results based on cannibalization provide an upper bound for system availability when cannibalization is not used. A lower bound may be derived by computing backorders B based on cannibalization and then assuming backorders are distributed randomly among systems. While system availability is highest under cannibalization, total failure generation is therefore highest and so total component backorders are highest.

In computing B_I under the assumption of random distribution of backorders the conditional probability $Pr(B_I = j|B = x)$ can be approximated as the chance of j successes from a binomial distribution with population parameter x, and chance of success of 1/n. $Pr(B_I = j|B = x)$ can also be computed more exactly, recognizing there can be a maximum of c backorders from any given system by combinatorial analysis as shown in Appendix II.

2.2 <u>Formulas When MTI ≠ o</u>

<u>One System Supported</u>. If s = o, the MTI may simply be added to the

resupply time, μ. As a general procedure this is erroneous. For example, it implies that so long as a spare is always available, the system will never go down, which is not true. It can fail during installation.

A different approach is suggested for constant MTI:

$$(2.2.1) \qquad U = \sum_{j=o}^{c} Pr(B=j)[1 - F(c-1-j; (\lambda) (MTI))]$$

The basis for this equation is that the number of components not functioning at some time t equals the number backordered at time t - MTI, plus the number which fail subsequently, failures being exponential as long as the system is up.

The sources of error in use of equation (2.2.1) are:

a. Calculation of the distribution on B by the M/M/C+S/C+S queue, section 2.1, will no longer be exact because of the impact of MTI on failure generations.

b. Calculation of failure rate in the interval (t-MTI, MTI] does not allow for the possible impact of components being installed at time t-MTI. They may cause the system to temporarily go down in the interval, before c-j component failures are accumulated.

More generally, when multi-component (type) systems are considered, it becomes necessary to account for the impact on demand rate of component type A of system down time caused by failure of component types B, C..... Barlow and Proschan [1] offer exact results in the context of systems without redundancy or spare pool. It seems likely to us that in the problems of the nature considered here, attention will have to focus on approximations based on adjustment of λ for down time, e.g. by setting $\lambda' = (\lambda)(A)$, where A is the system availability to be achieved.

Multiple Systems Supported. Equation (2.2.1) is formally generalized to

$$(2.2.1') \qquad U = \sum_{j=o}^{c} Pr(B_I = j)[1 - F (c-1-j; (\lambda)(MTI)]$$

No new sources of error are introduced by the need to compute the conditional distribution of B_I given B. Under cannibalization, for example,

for $n = 5$, $c = 2$,

$Pr(B_I = 1|B=1) = 1/5$ (one system has 1)

$Pr(B_I = 1|B=7) = 3/5$ (the first 5 backorders are distributed one to a system, and the next two are given to 2 of the systems leaving three with 1 backorder)

Under a smart policy the conditionals follow directly from the elaborated state spare.

III. WARM STANDBY

3.1 Exact Formulas When MTI = 0

Results obtained for cold standby, with n=1 or with n > 1 and cannibalization, hold with only a redefinition in λ_k required:

$$(3.1.1) \qquad \lambda_k = nc\lambda \qquad \text{for } k = o \text{ to } s$$
$$\lambda_k = [nc - (k-s)]\lambda \qquad \text{for } k > s$$

Similarly, although issue policy does not affect total backorders, as it did under cold standby, the same elaboration of the state space is still required to treat n > 1 and the smart issue policy - in order to infer system availability from system backorders.

Under FIFO issue policy, in which backorders are eliminated in the order in which they occur, no elaboration of state space is required. Since this is mathematically so convenient, and since FIFO is realistic in some situations, we will consider this case further.

To derive individual system backorders from total backorders - total backorders are found as with cannibalization - set

$$(3.1.2) \qquad Pr(B_I = c \mid B = j) = \prod_{i=1}^{c} \frac{j - (i-1)}{nc - (i-1)}$$

This is an application of the theory of sampling without replacement. Conceptually number the components from 1 to nc, with 1 to c being on system I. The probability that component 1 is on backorder given B = j is j/nc. The probability component 2 is on backorder given component 1 was, is (j-1)/(nc-1), etc. Underlying this is that each of the nc components is equally likely to be on backorder, and that knowing how j-i of the backorders are distributed amongst systems tells us nothing about the distribution of the remaining i backorders.

3.2 Formulas When MTI ≠ 0: Analogue to Cold Standby

Results obtained for cold standby hold, provided we substitute the appropriate expression for number of component failures in an interval equal

to MTI. The analogue to (2.2.1') is:

$$(3.2.1) \qquad U = \sum_{j=o}^{c} \Pr(B_I = j)[1 - F(0;(\lambda)(MTI)]^{c-j}$$

Sources of error, corresponding to those under cold standby are:

a. Calculation of distribution on B, from which B_I is determined, is no longer exact.

b. Some of the c-j non backordered components may be in the process of installation at time t-MTI.

The most reasonable adjustment on λ to account for the impact of MTI on failure generation is to set:

$$(3.2.1) \qquad \lambda' = (\lambda) \frac{MTBF}{MTBF+MTI} = (\lambda) \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + MTI}$$

i.e. multiply by the percent of time the component is working, given that it is not on backorder.

Under FIFO, calculation of the general distribution on B_I is done by a more complex version of equation (3.1.2).

$$(3.2.2) \qquad \Pr(B_I = m|B=j) = \binom{c}{m} \prod_{i=1}^{m} \frac{(j-(i-1))}{(nc-(i-1))} \prod_{i=m+1}^{c} \frac{nc - (i-1) - (j-m)}{nc - (i-1)}$$

Again, we are sampling without replacement. Ignoring $\binom{c}{m}$ for a moment, the rest of the expression is the probability that components 1 to m are on backorder, and m + 1 to c are not. $\binom{c}{m}$ is the number of ways of dividing the c components into subsets of size m (backordered) and size c-m (not backordered).

3.3 Formulas When MTI ≠ 0: Alternative Approach

This approach utilizes the perspective of renewal theory to relax the requirement that MTI be deterministic. Consider the state of an individual component "slot" on a system. A new cycle begins whenever a component is installed in the slot, and the process then passes through 3 states:

MTBF	MLDT	MTI
Installed	Backordered	Installation Underway

The middle state may be zero in some cycles; recall also that the system may be up even though a particular component is in the backorder or installation states.

Provided the time in each state is independent of what transpired in previous cycles, it is easily shown that installations comprise the renewal events of a renewal process, and that the steady state percentages of time in each state, or probability of being in each state, are equal to the mean time in a state divided by mean time between renewals (cf Ross, Renewal Reward process).

Now consider the same process where we "blot out" time spent on back-order. Clearly, the percentage of time in the installation state, given that a slot is not in the backordered state is MTI/(MTI + MTBF).

Defining L_I as the number of components of system I in installation, this suggests calculating

$$(3.3.1) \qquad U = \sum_{j=o}^{c} Pr(B_I=j)\ Pr(L_I=c-j)$$

$$= \sum_{j=o}^{c} Pr(B_I=j)\ \left(\frac{MTI}{MTI + MTBF}\right)^{c-j}$$

Unlike equation (3.2.1) we are interested in B_I at time t, not t-MTI, and we need make no assumption about the distribution of MTI, other than it be independent for each component.

This approach does not alter the problem of estimating the distribution on B when MTI $\neq$ 0. Furthermore, it is not entirely correct to assume, as the approach does, that the probabilities of being in the installation state are independent among components.

Dependence is occasioned by the dependency amongst the backorder states. (cf section 3.5)

3.4 Formulas When MTI $\neq$ 0: Comparison of Approaches

Recognizing that both approaches are approximations, they should still

closely agree for deterministic MTI. We note that by Taylor's expansion,

$$1 - F\,[0;(\lambda)(\mathrm{MTI})] = 1 - e^{-(\lambda)(\mathrm{MTI})} = (\lambda)(\mathrm{MTI}) - \frac{(\lambda)^2(\mathrm{MTI})^2}{2} + \text{Remainder Term}$$

and that by algebra and definition of MTBF in terms of λ,

$$\frac{\mathrm{MTI}}{\mathrm{MTI} + \mathrm{MTBF}} = (\lambda)(\mathrm{MTI}) - \frac{(\lambda^2)(\mathrm{MTI})^2}{(\lambda)(\mathrm{MTI}) + 1}$$

Thus the difference is a function of $(\lambda)^2(\mathrm{MTI})^2$ and higher order terms.

3.5 Approach Based on Independence

It is very convenient to treat warm standby as does the ACCLOGTROM model [3] by calculating average logistics down time and then assuming complete independence between components of what state they are in.

Under the assumption, with components in parallel,

$$(3.5.1) \qquad U = \left(\frac{\mathrm{MLDT} + \mathrm{MTI}}{\mathrm{MTBF} + \mathrm{MLDT} + \mathrm{MTI}}\right)^{c}$$

This approach was tested in two cases where exact answers were available and was found disastrous (Appendix III). MTI was 0 in both cases, permitting an exact calculation, and n was 1.

As a verification of (3.3.1) we have shown that it leads to the same results as 3.5.1, provided B is taken to be binomially distributed, with population parameter nc; i.e., provided the independence assumption were correct.

4. Conclusion

We have shown how to obtain exact solutions for steady state availability when installation time is 0. These solutions depend on the issue policy of the pool manager, and whether the system is operating under warm standby or cold standby.

Approximations are provided when installation time is not 0. The major problem to be solved in refining these approximations is how to account for

the interaction between installation time and failure generation.

While the problem addressed was based on a real world problem description related to electronic systems, it is easy to imagine some variations in the assumptions which might need to be treated in other contexts; e.g., diagnostic time is significant, all redundant components must be installed at one time, and so on.

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APPENDIX I

SOLUTION OF AUGMENTED DIFFERENCE EQUATIONS

The difference equations up to state p_s follow a normal birth/death process; e.g.

(A1.1) $$0 = -(2\lambda + su)\, p_s + (s+1)(\mu)\, p_{s+1} + (2\lambda) p_{s-1}$$

Hence, for k = 1 - > s+1,

(A1.2) $$p_k = p_o \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}$$

$$\lambda_i = 2\lambda$$

$$\mu_i = (i)(\mu)$$

We can also write these difference equations:

(A1.3a) $$0 = -[(s+3)(\mu) + \lambda]\, p_{s+3} + (s+4)(\mu)\, p_{s+4}$$

$$+ (\lambda)\, p_{s+2,a} + (2\lambda)\, p_{s+2,b}$$

(A1.3b) $$0 = -(s+4)(\mu)\, p_{s+4} + \lambda\, p_{s+3}$$

Substituting for p_{s+4} from (3b) into (3a)

(A1.4) $$0 = -(s+3)\mu p_{s+3} + \lambda p_{s+2,a} + 2\lambda\, p_{s+2,b}$$

Substituting for $p_{s+2,a}$ from (2.1.3a) into (A1.4)

(A1.5) $$0 = -(s+3)(\mu)\, p_{s+3} + \frac{(\lambda)(\lambda)}{\lambda + (s+2)\mu}\, p_{s+1}$$

$$+ (2\lambda) p_{s+2,b}$$

Comparing (A1.5) and (2.1.3b),

$$\text{(A1.6)} \quad -\,(2\lambda + (s+2)(\mu)]\, p_{s+2,b} + \lambda\, p_{s+1} = + \frac{(\lambda)(\lambda)}{\lambda + (s+2)(\mu)}\, p_{s+1} + (2\lambda)\, p_{s+2,b}$$

From (A1.6) we can solve for $p_{s+2,b}$ in terms of p_{s+1}, which is obtained in (A1.2), and then use this result and (A1.5) to get p_{s+3}.

APPENDIX II

A PROBLEM IN COMBINATORIAL MATHEMATICS

Problem

Each of x backorders is assigned to one of n systems. We imagine that the backorders are numbered from 1 to x, and that the assignments are made sequentially in time. We are interested in the probability that System I, chosen at random, has j backorders assigned. The assignment mechanism is: assign backorders with equal probability to all feasible systems, where a feasible system is one with $< c$ backorders already assigned.

General Procedure

We find the number of (assumed equally likely) assignment patterns, where we do not require that system I have j backorders. Then we find the number given this restriction, and take the ratio. A pattern is distinguished not only by the number of backorders for each system, but by which systems are assigned to which backorders, i.e., both backorders and systems are distinguished.

We give the details while illustrating with an example for which $n = 5$, $c = 2$, $x = 4$, and $j = 1$.

Find Unrestricted Number of Patterns:

(1) Identify all possible groupings of backorders among systems, treating backorders and systems as indistinguishable. Begin with groupings using the least possible number of systems. In our example, groupings are:

2-2-0-0-0 (2 systems have 2 backorders each)

2-1-1-0-0

1-1-1-1-0

(2) For each grouping number of distinguishable patterns is found by first treating backorders as indistinguishable, then correcting. For our example:

Feasible Grouping	Backorders Indistinguishable		Correction	Total
2-2-0-0-0	$\binom{5}{2}$	x	$\binom{4}{2}$	60
2-1-1-0-0	$(5)\binom{4}{2}$	x	$\binom{4}{2}\binom{2}{1}$	360
1-1-1-1-0	$\binom{5}{2}$	x	$\binom{4}{1}\binom{3}{1}\binom{2}{1}$	120
				540

We illustrate with grouping 2-1-1. There are five ways to pick the system with 2 backorders. There are $\binom{4}{2}$ ways to pick which 2 of the remaining 4 systems has 2 backorders. There are $\binom{4}{2}$ ways to choose which of the 4 backorders, now treated as distinguishable, go with the system with 2 backorders. There are $\binom{2}{1}$ ways to choose which of the two remaining unassigned backorders go with the first of the remaining two unassigned systems.

Find Conditional Number of Patterns:

Feasible Grouping	Backorders Indistinguishable		Correction	Total
2-1-1-0-0	(4) (3)	x	$\binom{4}{2}\binom{2}{1}$	144
1-1-1-1-0	$\binom{4}{3}$		$\binom{4}{1}\binom{3}{1}\binom{2}{1}$	96
				240

We illustrate with grouping 2 - 1 - 1. We know system I must account for one of the systems with 1 backorder. There are 4 ways, therefore, to pick the system with 2 backorders, and 3 systems to choose from to be the other system with 1 backorder. Correction logic is always as for the unrestricted case.

$$\text{Probability} = \frac{240}{540} = .44$$

$$\text{Corresponding Binomial is } \binom{4}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{3} = .41$$

<u>Source of Error</u>

Certain patterns are actually more likely than others due to the constraint on total backorders from any one system. Consider again the case where n=5, c=2 and x=4, and label the systems a,b,c,d,e so that (a,b,c,-) represents the assignment of the first three backorders to systems a,b,c respectively, with the fourth as yet unassigned. Then (a,b,c,-) and (a,b,b,-) are equally likely by usual multinomial arguments. However, (a,b,b,d) is more likely than (a,b,c,d) since given (a,b,b,-) the fourth backorder must be a,c,d or e with equal probability, while given (a,b,c,-), the fourth backorder can be any of a,b,c,d or e.

APPENDIX III

EVALUATION OF INDEPENDENCE ASSUMPTION

The case of warm standby, n = 1 and c = 2, was run for s = 0 and 1, and a range of λ. MTI was 0. The table below shows the exact unavailability, and the ratio of the estimates based on independence to this exact value. To calculate the independence based estimate, the following equivalance* is used:

$$\frac{LDT}{LDT+MTBF} = \frac{E_x(B)}{nc}$$

As can be seen from the table below, unavailability based on independence is quite low for s = 1, but exact for s = 0. When s = 0, there is in fact independence between component states as resupply time does not depend on the stock position of a supplier.

λ	Unavailabilities		Unavailability Ratios	
	s = 0	s = 1	s = 0	s = 1
.010	.010%	.000%	1.00	.01
.021	.041%	.001%	1.00	.03
.043	.170%	.005%	1.00	.06
.089	.670%	.040%	1.00	.13
.185	2.43%	.292%	1.00	.24
.383	7.68%	1.79%	1.00	.43
.795	19.6%	8.00%	1.00	.67

*Bernard Price has developed a simple proof of this equivalence (unpublished notes).

APPENDIX IV

INFINITE SOURCE APPROXIMATIONS

Infinite source estimates of unavailability are calculated by basing total backorders on the Poisson. The implication of infinite source is that the failure rate does not depend on the number in repair.

Cold Standby: $Pr(B=j) = p(j+s;n\lambda')$ where $p(x;y)$ is Poisson function

Warm Standby: $Pr(B=j) = p(j+s;nc\lambda')$

λ' was either set to λ or based on exact estimates of system down time.

Four cases were tried to probe the accuracy of infinite source approximations:

Cold Standby: $n = 1$; $c = 2$; $s = 0$ and $s = 1$.

Warm Standby: $n = 1$; $c = 2$; $s = 0$ and $s = 1$.

Single system cases were chosen because they may be most common when built in redundancy is extensive, they are easiest to model, and one might expect infinite source approximations to work worst when $n = 1$. MTI was set to 0 for simplicity and to permit exact answers for comparison.

When λ was adjusted, it was done as follows:

Cold Standby: $\lambda' = (\lambda)(1-U)$

Warm Standby: $$\lambda' = (\lambda)\frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + MLDT}$$

$MLDT = 1 \qquad s = 0$

$$MLDT = \frac{E(B)}{E(D)} \qquad s > 0$$

$E(D)$ = expected failures per unit time

$$= 2\lambda * [1 - U - \frac{1}{2} Pr(B=1)]$$

Exact values of U, Pr(B=1) were used.

Included in this Appendix are three tables showing, respectively: exact unavailabilities, ratios of unavailabilities calculated by simple Poisson to exact, ratio of expected backorders calculated by simple Poisson to exact.

Reviewing the tables, it is apparent the infinite source approximation breaks down under warm standby, calculation of unavailability, even though calculation of expected backorders is reasonably good. Adjusted λ helps somewhat. Under cold standby, there is no problem so long as system reliability is high, and adjusting λ, in a simple fashion, does not help.

A rationalization of the results is that under cold standby demand stops when there are two backorders, but so long as unavavailability is low, even the infinite source calculation expects little chance of greater than 2 backorders. Under warm standby chances of the 2 backorders needed to cause system unavailability are greatly reduced in the finite source model by the halving of the demand rate once there is one backorder. The infinite source approximation does not capture this effect. Its estimate of expected backorders are not bad because this value is dominated by the probability of 1 backorder, i.e. two backorders occurs relatively infrequently.

Unavailabilities

λ	Cold Standby		Warm Standby	
	s=0	s=1	s=0	s=1
.010	.005%	.000	.010	.000
.021	.021%	.000	.041	.001
.043	.089%	.001	.170	.005
.089	.364%	.011	.670	.040
.185	1.42%	.088	2.43	.292
.383	5.04%	.640	7.68	1.79
.795	15.0%	3.82	19.6	8.00

TABLE 1

UNAVAILABILITIES

λ	Ratios Unadjusted				Ratios Adjusted			
	Cold Standby		Warm Standby		Cold Standby		Warm Standby	
	s=0	s=1	s=0	s=1	s=0	s=1	s=0	s=1
.010	1.00	1.00	2.01	2.01	1.00	1.00	1.97	2.01
.021	1.01	1.01	2.03	2.02	1.01	1.00	1.95	2.02
.043	1.01	1.01	2.05	2.04	1.01	1.01	1.89	2.03
.089	1.03	1.02	2.11	2.09	1.02	1.01	1.79	2.05
.185	1.06	1.05	2.20	2.19	1.04	1.01	1.63	2.04
.383	1.13	1.10	2.33	2.39	1.03	.96	1.39	1.90
.795	1.27	1.22	2.41	2.68	.985	.82	1.13	1.37

TABLE 2

UNAVAILABILITY RATES

	Ratios Unadjusted				Ratios Adjusted			
λ	Cold Standby		Warm Standby		Cold Standay		Warm Standby	
	s=0	s=1	s=0	s=1	s=0	s=1	s=0	s=1
.010	1.00	1.00	1.01	1.01	1.00	1.00	1.00	1.01
.021	1.00	1.00	1.02	1.01	1.00	1.00	1.00	1.01
.043	1.00	1.00	1.04	1.03	1.00	1.00	1.00	1.03
.089	1.00	1.00	1.09	1.06	1.00	.99	1.00	1.05
.185	1.01	1.01	1.18	1.14	1.00	.98	1.00	1.08
.383	1.05	1.03	1.38	1.31	1.00	.94	1.00	1.12
.795	1.18	1.12	1.79	1.72	1.00	.84	1.00	1.12

TABLE 3

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	Ratios Unadjusted				Ratios Adjusted			
λ	Cold Standby		Warm Standby		Cold Standay		Warm Standby	
	s=0	s=1	s=0	s=1	s=0	s=1	s=0	s=1
.010	1.00	1.00	1.01	1.01	1.00	1.00	1.00	1.01
.021	1.00	1.00	1.02	1.01	1.00	1.00	1.00	1.01
.043	1.00	1.00	1.04	1.03	1.00	1.00	1.00	1.03
.089	1.00	1.00	1.09	1.06	1.00	.99	1.00	1.05
.185	1.01	1.01	1.18	1.14	1.00	.98	1.00	1.08
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	Ratios Unadjusted				Ratios Adjusted			
λ	Cold Standby		Warm Standby		Cold Standay		Warm Standby	
	s=0	s=1	s=0	s=1	s=0	s=1	s=0	s=1
.010	1.00	1.00	1.01	1.01	1.00	1.00	1.00	1.01
.021	1.00	1.00	1.02	1.01	1.00	1.00	1.00	1.01
.043	1.00	1.00	1.04	1.03	1.00	1.00	1.00	1.03
.089	1.00	1.00	1.09	1.06	1.00	.99	1.00	1.05
.185	1.01	1.01	1.18	1.14	1.00	.98	1.00	1.08
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